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Annamaria Sorato and Bruno Viscolani

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Modelling smoothly the joint effect of several advertising media on sales in a homogeneous market *

ANNAMARIA SORATO

<amsorato@math.unive.it>

Dept. of Applied Mathematics

University of Venice

Dorsoduro 3825/E, I-30123 Venezia, Italy

BRUNO VISCOLANI

<viscolani@math.unipd.it>

Dept. of Pure and Applied Mathematics

University of Padua

Via Trieste 63, I-35121 Padova, Italy

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Abstract. Decision on the use of different advertising media is a critical issue in marketing. Drawing on some literature related to the dynamic Nerlove-Arrow model, we propose a non-linear programming framework for discussing how different advertising media may jointly affect the demand for a good. Starting from the idea that different advertising efforts may not simply add (linearly) to produce the demand result, we examine a few special media combination mechanisms which can be represented by smooth functions.

Keywords: Marketing; Advertising; Production; Nonlinear programming.

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Correspondence to:

Annamaria Sorato Dept. of Applied Mathematics, University of Venice

Dorsoduro 3825/e

30123 Venezia, Italy

Phone: ++39 (041)-234-6920

Fax: ++39 (041)-522-1756

E-mail: amsorato@unive.it

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1 Introduction

Kotler and Trias de Bes in the opening of “Lateral Marketing: New Techniques for Finding Breakthrough Ideas”, notice that “reaching success at the beginning of the twenty-first century is more difficult” [11, p. 3], and they suggest that one reason is that “audiences are so diverse in their media habits that companies have to invest in many media to reach them” [11, p. 11]. Therefore, advertising for the launch of new products, for example, is more and more expensive. The choice of a firm advertising effort in a given period while using a set of different media is crucial and must take into account different features of the joint use of media. This leads to so called integrated marketing communication, which “emphasises the benefits of harnessing synergy across multiple media to build brand equity of products and services” [14]. Then the problem of allocating marketing resources to multiple activities, in particular different media, is a relevant problem [15].

The joint impact of multiple advertising activities (e.g. television and print advertising) creates added value, a phenomenon which is known as synergy [14]. At a first approximation level, one may consider simple superposition of the effects of different advertising activities. We may also observe advertising saturation [11, p. 11], which lowers communication effectiveness, and even negative interference among advertising messages. Additivity of advertising efforts is a customary assumption in game theoretic literature, where different advertising strategies are chosen by different agents (see e.g. [10], [9, pages 103, 108, 119], [18], [22]). Additivity, there, is chosen mainly to obtain tractable models. The typical situation concerns a distribution channel, where two agents (a manufacturer and a retailer) advertise the same product using different means, in order to obtain higher profits.

Here, we want to model the joint effect of several advertising media from a wide viewpoint and discuss a consequent, rather general profit problem. We choose a static framework, and hence obtain a nonlinear programming problem statement. The study should pave the way to tackle, on the one hand, dynamic profit problems with several advertising media and, on the other hand, static or dynamic game theoretic analysis of distribution channel problems.

A first paper addressing marketing decisions as a nonlinear programming problem is due to Dorfman and Steiner [4]. Recently, an analogous problem has been discussed in [21], where the focus is on the market heterogeneity. On the other hand, static games have been proposed in marketing contexts (e.g. [10], [18] and [22]) and, in particular, Schoonbeek and Kooreman [18] stress the importance of static modelling of advertising decision making. Essentially, the choice of a static framework is reasonable in those situations in which either goodwill depreciates rapidly see e.g. [3], or the time horizon is short.

Our problem is essentially a case of single-output profit maximization in the *Theory of the Firm* (see e.g. [13, Chapter 5, Production]): advertising efforts using different media are the “inputs”, actual demand for the good is the “output”, sale of the good in the market determines revenue, and the firm wants to maximize profit. We focus on the transformation of inputs into output, i.e. on the way advertising efforts determine demand for the good.

In the paper [21] the advertising efforts combination is represented in the simplest way, by means of a linear model, and the focus is on the market heterogeneity. Such a linearity assumption is essentially the same as adopted to model the joint effect of the advertising

efforts of different (usually two) decision makers in a distribution channel (see [7], [6], [8] for dynamic models, and [9, p. 103], [22] for static ones). Here we consider a homogeneous market, instead of a heterogeneous one, in order to focus on the combination mechanism of the different advertising efforts.

The paper is organized as follows. In Section 2 we introduce the model and the general problem. In Section 3 we characterize the optimal solutions of the profit problem. In Section 4 we analyze the problem and its solutions for some special advertising productivity models.

2 Advertising, goodwill, demand

Let G be the stock of goodwill of the good, and let the demand for the good be an increasing and concave function of G ,

$$D = f(G), \quad (1)$$

where $f(G) \geq 0$, $f'(G) > 0$, $f''(G) \leq 0$. This is in agreement with similar assumptions on demand rates in some known dynamic models (see e.g. [16], [5], [19]).

The goodwill G is the result of the firm advertising activity, which involves n different advertising media. For all $i = 1, \dots, n$, we denote by $u_i \geq 0$ the advertising effort using the i -th medium and we assume that the effect of the joint media use on goodwill is represented by

$$G = \Phi(u) + G^0 = \Phi(u_1, u_2, \dots, u_n) + G^0, \quad (2)$$

where the *advertising productivity* function $\Phi : [0, +\infty)^n \rightarrow [0, +\infty)$ is a continuously differentiable functions, increasing and such that $\Phi(0) = 0$, whereas G^0 is the *no-ad* goodwill.

The simplest example is the *linear advertising* model

$$\Phi(u) = \sum_{i=1}^n \gamma_i u_i, \quad (3)$$

where $\gamma_i > 0$ is the marginal productivity of the advertising effort using the i th medium, in terms of goodwill, $i = 1, \dots, n$. Here the effect of the joint media activation on goodwill is the sum of the media separate effects.

The special model (3) has been used in [21], where an advertising process toward a segmented market was considered. Moreover, (3) is in agreement with some assumptions adopted to model the manufacturer-retailer interaction, in the framework of dynamic game theory (see [7], [6], [8]), and of static game theory (see [9, p. 103]).

Nevertheless, it is not obvious that the linear model (3) should provide a faithful representation of a market reaction to advertising. At a higher (representation) flexibility level we have the *nonlinear additive advertising* model

$$\Phi(u) = \sum_{i=1}^n \varphi_i(u_i), \quad (4)$$

where for all $i = 1, \dots, n$, $\varphi_i : [0, +\infty) \rightarrow [0, +\infty)$ is increasing, concave and such that $\varphi_i(0) = 0$; moreover, assuming that φ_i is twice continuously differentiable, we require that $\varphi_i'(u_i) > 0$, and $\varphi_i''(u_i) \leq 0$.

An example of non-additive advertising model is provided by [2], and documented also in [9, p. 106]: it assumes a polynomial representation of the joint effect of different advertising efforts on sales.

We want to analyze the problem of maximizing the firm profit

$$\Pi(u) = \pi f(G) - \sum_{i=1}^n c_i(u_i), \quad (5)$$

where $\pi > 0$ is the marginal profit of goodwill, gross of advertising costs, and $c_i(u_i)$ is the cost associated with the i th medium level $u_i \geq 0$. We assume that $c_i(u_i)$ is an increasing, convex, and continuously differentiable function, where $c_i(0) = 0$, $c_i'(u_i) \geq 0$, $c_i''(u_i) > 0$, and $\lim_{u_i \rightarrow \infty} c_i'(u_i) = +\infty$.

We first consider the important case of a model with a twice continuously differentiable function Φ , obtaining the characterization of optimal solutions to the profit problem; we go into details of the special *linear* and *nonlinear additive advertising* models (3), (4); we introduce and analyze a *nonlinear multiplicative advertising* model.

3 The profit problem

Here we assume that the advertising productivity function $\Phi(u)$ is continuous in the domain $[0, +\infty)^n$ and twice continuously differentiable in $(0, +\infty)^n$, moreover we require that $\partial\Phi/\partial u_i > 0$, $i = 1, \dots, n$.

3.1 Optimal advertising effort

Theorem 1 *If u^* is an optimal solution with associated goodwill level G^* , then, for all $i = 1, \dots, n$,*

- *either $u_i^* > 0$ and*

$$\pi f'(G^*) \frac{\partial\Phi(u^*)}{\partial u_i} = c_i'(u_i^*), \quad (6)$$

- *or $u_i^* = 0$ and*

$$\pi f'(G^*) \frac{\partial\Phi(u^*)}{\partial u_i} \leq c_i'(0). \quad (7)$$

PROOF: The profit $\Pi(u)$ is a continuously differentiable function and its partial derivative with respect to u_i is

$$\frac{\partial\Pi(u)}{\partial u_i} = \pi f'(G) \frac{\partial\Phi(u)}{\partial u_i} - c_i'(u_i). \quad (8)$$

Then the conditions (6) and (7) are the first order necessary conditions for the i th coordinate of a maximum point (see e.g. [12, p. 314]) in the two distinct cases that the i th medium is really used or not. \square

Remark: It may occur that an optimal solution has no zero component, or some zero components, or even is the zero vector. For the last case to occur, i.e. for the decision of “not activating” any advertising medium to be optimal, it is necessary that

$$\pi f'(G^0) \frac{\partial \Phi(0)}{\partial u_i} \leq c'_i(0), \quad \text{for all } i. \quad (9)$$

This condition might be satisfied in case of a large no-ad goodwill G^0 and strictly positive marginal advertising costs at advertising level 0.

Theorem 2 *If the function $\Phi(u)$ is concave, then a point u^* satisfying the necessary conditions of Theorem 1 is an optimal solution.*

PROOF: Let $\Phi(u)$ be concave, so that its Hessian matrix $\nabla^2 \Phi(u)$ is negative semidefinite. The Hessian matrix of the profit function is

$$\begin{aligned} \nabla^2 \Pi(u) = & \pi f''(G) (\nabla \Phi(u))^T (\nabla \Phi(u)) + \pi f'(G) \nabla^2 \Phi(u) \\ & - \text{diag}(c''_1(u_1), c''_2(u_2), \dots, c''_n(u_n)), \end{aligned} \quad (10)$$

which is negative semidefinite as the sum of three matrices of this kind. Hence the profit function $\Pi(u)$ is concave, which makes the necessary conditions also sufficient for the optimum (see e.g. [12, p. 181]). \square

Theorem 3 *If the function $\Phi(u)$ is monotonically increasing and concave, then there exists an optimal solution u^* to the profit problem.*

PROOF: The hypothesis states that $\nabla \Phi(u) \geq 0$ and that $\nabla^2 \Phi(u)$ is negative semidefinite.

If $\Pi(u) \leq 0$ for all $u \geq 0$, then $u^* = 0$ is an optimal solution.

Alternatively, let $\Pi(u) > 0$ at some u .

For all $u \geq 0$, $u \neq 0$, and $t > 0$ the following inequality holds

$$\begin{aligned} \frac{d}{dt} \Pi(tu) &= \pi f'(G) \nabla \Phi(tu) u - \sum_{i=1}^n c'_i(tu_i) u_i \\ &\leq \psi(t, u) = \pi f'(G^0) \nabla \Phi(tu) u - \sum_{i=1}^n c'_i(tu_i) u_i, \end{aligned}$$

because $G = \Phi(tu) + G^0 \geq G^0$, $f'(\cdot)$ is a decreasing function and $\pi \nabla \Phi(tu) u > 0$. The upper bound $\psi(t, u)$ of $d\Pi(tu)/dt$ is a decreasing function of t , for all u , because

$$\frac{\partial \psi(t, u)}{\partial t} = \pi f'(G^0) u' \nabla^2 \Phi(tu) u - \sum_{i=1}^n c''_i(tu_i) u_i^2 < 0, \quad (11)$$

where $\nabla^2 \Phi(\cdot)$ is negative semidefinite and $c''_i(\cdot) > 0$. Moreover, $\psi(t, u) < 0$ for sufficiently large t , because $\lim_{u_i \rightarrow \infty} c'_i(u_i) = +\infty$. Hence, the equation

$$\psi(t, u) = 0$$

defines implicitly a function $\hat{t} : [0, +\infty)^n \setminus \{0\} \rightarrow [0, +\infty)$ such that $\psi(\hat{t}(u), u) = 0$ for all u . Moreover, the assumptions of the implicit function Theorem (see [17, p. 224]) are satisfied, because of (11), and $\hat{t}(u)$ is a differentiable function. We observe that

$$\Pi(tu) \leq \Pi(\hat{t}(u)u), \quad t \geq \hat{t}(u),$$

and in particular this is true for all vectors u with unit norm. Therefore, a point u^* is an optimal solution of the profit problem if and only if $u^* = tu$, with (t, u) optimal solution of the following problem,

$$\begin{aligned} \max \quad & \Pi(tu), \\ \text{s.t.} \quad & t \geq 0, \quad u \in [0, +\infty)^n, \\ & \|u\| = 1, \\ & t - \hat{t}(u) \leq 0, \end{aligned}$$

which has a compact feasible set and a continuous objective function. Therefore there exists a maximum point of the profit Π . \square

4 Special advertising productivity models

Here we consider two main ways to model advertising productivity: the additive and the nonlinear multiplicative.

4.1 Additive advertising model

In the case of the additive advertising model (4), where $\Phi(u) = \sum_{i=1}^n \varphi_i(u_i)$, with the assumptions stated in Section 2, the function $\Phi(\cdot)$ is twice continuously differentiable, increasing and concave; therefore, from Theorems 1-3 we have that there exists an optimal solution u^* which determines the goodwill value $G^* = \Phi(u^*) + G^0$, such that for all $i = 1, \dots, n$,

- either $u_i^* > 0$ and

$$\pi f'(G^*) \varphi'_i(u_i^*) = c'_i(u_i^*), \quad (12)$$

- or $u_i^* = 0$ and

$$\pi f'(G^*) \varphi'_i(0) \leq c'_i(0). \quad (13)$$

The linear advertising model (3) is a special case in which $\varphi'_i = \gamma_i$; it is the homogeneous market case of the model discussed in [21].

In the following examples we propose the analysis of some special instances of the problem under the additive advertising assumption.

EXAMPLE 1 Assume that the cost functions are

$$c_i(u_i) = \frac{1}{2} k_i u_i^2, \quad i = 1, \dots, n, \quad (14)$$

where $k_i > 0$. Now the condition (13) is equivalent to $\pi f'(G^*)\varphi'_i(0) \leq 0$, which contradicts our general assumption that $f' > 0$ and $\varphi'_i > 0$; therefore any optimal solution u^* must be positive and, for all i , equation (12) must hold, i.e.

$$\pi f'(\Phi(u) + G^0)\varphi'_i(u_i) = k_i u_i, \quad i = 1, \dots, n. \quad (15)$$

From the implicit function Theorem (see [17, p. 224]) we obtain that u^* is a differentiable function of $(k_1, \dots, k_n, \pi, G^0)$.

EXAMPLE 1.1 Assume $c_i(u_i) = \frac{1}{2}k_i u_i^2$, $i = 1, \dots, n$, and linear demand

$$f(G) = \beta G, \quad (16)$$

where $\beta > 0$. The optimality conditions (15) become

$$\pi\beta\varphi'_i(u_i) = k_i u_i, \quad i = 1, \dots, n, \quad (17)$$

and are n independent equations. As $\varphi'_i(u_i) > 0$ and $\varphi''_i(u_i) \leq 0$, the i th condition has a unique solution $u_i^* > 0$. Therefore there exists a unique solution $u^* > 0$ to the profit problem. Clearly the solution does not depend on the no-ad goodwill G^0 , as this parameter does not enter the optimality conditions. On the other hand, from the implicit function Theorem we obtain that u_i^* is a differentiable function of (k_i, π, β) , it is decreasing in the cost factor k_i , and increasing in the profit margin and goodwill efficiency parameters π, β .

As a special instance (see [18] for a use of it), let

$$\varphi_i(u_i) = \gamma_i \sqrt{u_i}, \quad i = 1, \dots, n, \quad (18)$$

where $\gamma_i > 0$. Then the equation (17) reads

$$\frac{\pi\beta\gamma_i}{2\sqrt{u_i}} = k_i u_i,$$

which gives

$$u_i^* = \left(\frac{\pi\beta\gamma_i}{2k_i} \right)^{2/3}.$$

We observe, of course, that the optimal advertising effort u_i^* is an increasing function of the effectiveness parameter γ_i .

As a second special instance, let us consider the linear advertising model (3), i.e. the additive model with

$$\varphi_i(u_i) = \gamma_i u_i, \quad i = 1, \dots, n,$$

where $\gamma_i > 0$. Then the equation (17) reads

$$\pi\beta\gamma_i = k_i u_i,$$

which gives

$$u_i^* = \frac{\pi\beta\gamma_i}{k_i}.$$

Again we notice that the optimal advertising effort u_i^* increases with γ_i .

EXAMPLE 1.2 Assume $c_i(u_i) = \frac{1}{2}k_i u_i^2$, $i = 1, \dots, n$, and logarithmic demand

$$f(G) = \beta \ln(1 + G),$$

where $\beta > 0$. The optimality conditions (15) become

$$\frac{\pi\beta\varphi'_i(u_i)}{1 + \sum_{j=1}^n \varphi_j(u_j) + G^0} = k_i u_i, \quad i = 1, \dots, n. \quad (19)$$

For any given values of u_j , $j \neq i$, the above condition has a unique solution $\hat{u}_i > 0$, because $\varphi'_i(u_i) > 0$ and $\varphi''_i(u_i) \leq 0$. Nevertheless, \hat{u}_i depends on the given values of u_j , $j \neq i$, and this may allow multiple solutions to the optimality conditions.

As a special instance, let us consider the linear advertising model (3), $\Phi(u) = \sum_{i=1}^n \gamma_i u_i$. Then the equation (19), with $\varphi'_i(u_i) = \gamma_i$, reads

$$\frac{\pi\beta\gamma_i}{1 + \sum_{j=1}^n \gamma_j u_j + G^0} = k_i u_i, \quad i = 1, \dots, n. \quad (20)$$

Here we observe that

$$u_i = \frac{\gamma_i}{k_i} \omega, \quad \omega = \frac{\pi\beta}{1 + \sum_{j=1}^n \gamma_j u_j + G^0},$$

so that (20) gives the unique (positive) solution

$$u_i^* = \frac{-(1 + G^0) + \sqrt{(1 + G^0)^2 + 4\pi\beta\Lambda}}{2\Lambda} \frac{\gamma_i}{k_i}, \quad i = 1, \dots, n,$$

$$\Lambda = \sum_{j=1}^n \frac{\gamma_j^2}{k_j}.$$

In the special symmetric case with the linear advertising model, i.e. with

$$\gamma_i = \gamma, \quad k_i = k, \quad i = 1, \dots, n,$$

we obtain

$$u_i^* = -\frac{1 + G^0}{2n\gamma} + \sqrt{\left(\frac{1 + G^0}{2n\gamma}\right)^2 + \frac{\pi\beta}{nk}}, \quad i = 1, \dots, n.$$

On the other hand, in a symmetric situation with a general nonlinear advertising model, i.e. with

$$\varphi_i(\cdot) = \varphi(\cdot), \quad k_i = k, \quad i = 1, \dots, n,$$

we obtain that a solution u must have equal components $u_i = z$, where z is the unique solution of the equation

$$\pi\beta\varphi'(z) = kz(1 + G^0 + n\varphi(z)).$$

4.2 Nonlinear multiplicative advertising model

We call *nonlinear multiplicative* the advertising model in which the goodwill is determined by the equation (2) with the advertising productivity function

$$\Phi(u) = \prod_{i=1}^n \varphi_i(u_i), \quad (21)$$

where for all $i = 1, \dots, n$, $\varphi_i : [0, +\infty) \rightarrow [0, +\infty)$ is increasing, concave and such that $\varphi_i(0) = \varphi_i^0 \geq 0$. We assume further that φ_i is twice continuously differentiable and we require that $\varphi_i'(u_i) > 0$ and $\varphi_i''(u_i) \leq 0$. Hence, the function $\Phi(\cdot)$ is twice continuously differentiable and increasing, whereas it is not concave in general.

Theorem 1 of Section 3.1 holds here and states that if the point u^* is an optimal solution, then for $G^* = \prod_{i=1}^n \varphi_i(u_i^*) + G^0$ and for all $i = 1, \dots, n$,

- either $u_i^* > 0$ and

$$\pi f'(G^*) \varphi_i'(u_i^*) \prod_{j \neq i} \varphi_j(u_j^*) = c_i'(u_i^*), \quad (22)$$

- or $u_i^* = 0$ and

$$\pi f'(G^*) \varphi_i'(0) \prod_{j \neq i} \varphi_j(u_j^*) \leq c_i'(0). \quad (23)$$

On the other hand, we cannot use Theorems 2-3 to state sufficiency of the conditions above nor existence of an optimal solution, because Φ may not be concave. In fact, we know from [1, p. 162, Corollary 5.18] that the function Φ , as defined by the product (21) under our assumptions, is semistrictly quasi-concave.

In the following subsections we consider two cases of nonlinear multiplicative advertising model with concave Φ .

4.2.1 Cobb–Douglas advertising model

We consider here the nonlinear multiplicative advertising model with *Cobb–Douglas* advertising productivity function,

$$\Phi(u) = \prod_{i=1}^n u_i^{\alpha_i}, \quad (24)$$

where $\alpha_i > 0$, $i = 1, \dots, n$, and $\sum_{i=1}^n \alpha_i \leq 1$. This case is particularly interesting, because the Cobb–Douglas function is a typical choice as production function in microeconomics, see e.g. [13, p. 130].

The function (24) is monotonically increasing, has the nonnegative orthant of \mathbb{R}^n as its domain, and in view of the above parameter conditions is a concave function (see [1, p. 161, Lemma 5.14]). Then Φ has decreasing or constant returns to scale, which is a reasonable feature of the advertising process. We notice that this would not be true if $\sum_{i=1}^n \alpha_i > 1$. As far as the optimization problem is concerned, Theorems 2-3 hold, in addition to Theorem 1,

and we have that there exists an optimal solution u^* which determines the goodwill value $G^* = \prod_{i=1}^n (u_i^*)^{\alpha_i} + G^0$, such that for all $i = 1, \dots, n$, either the condition (22) or (23) holds.

In this case, either all non-zero solutions $u \neq 0$ are too expensive and $u^* = 0$ is the unique optimal solution, or there exists an interior solution u^* to the optimality conditions, and then u^* is the unique optimal solution.

EXAMPLE 2 Assume quadratic costs, $c_i(u_i) = k_i u_i^2/2$, $i = 1, \dots, n$, linear demand, $f(G) = \beta G$, and the special Cobb–Douglas advertising model

$$\Phi(u) = \prod_{i=1}^n u_i^{1/n}. \quad (25)$$

Then the equations (22) become

$$\frac{\pi\beta}{n} u_i^{-(n-1)/n} \prod_{j \neq i} u_j^{1/n} = k_i u_i, \quad i = 1, \dots, n,$$

i.e., after setting $u_i = e^{y_i}$, the system of linear equations

$$(2n-1)y_i - \sum_{j \neq i} y_j = n \ln \left(\frac{\pi\beta}{nk_i} \right), \quad i = 1, \dots, n,$$

which has a unique solution, because its coefficient matrix is strictly diagonally dominant and then nonsingular (see [20, p. 6, Theorem 1.4]). In the case of two media, $n = 2$, we obtain the optimal solution

$$u_i^* = \frac{\pi\beta}{2} / \sqrt[4]{k_i^3 k_j}.$$

In the symmetric case with n media, i.e. when $k_i = k$ for all i , we obtain the optimal solution

$$u_i^* = \frac{\pi\beta}{nk}.$$

4.2.2 An enhanced advertising model: two media

We consider here the nonlinear multiplicative advertising model, representing a pair of media, with advertising productivity function

$$\Phi(u_1, u_2) = \varphi_1(u_1) \varphi_2(u_2), \quad (26)$$

where, for some $\gamma_1 > 0$ and $\alpha \in (0, 1)$,

$$\varphi_1(u_1) = \gamma_1 u_1^\alpha, \quad (27)$$

$$\varphi_2(u_2) = \left(\frac{1 + 2u_2}{1 + u_2} \right)^{1-\alpha}. \quad (28)$$

We observe that φ_1 and φ_2 are positive, twice differentiable, increasing and concave functions; moreover φ_2 is bounded,

$$\varphi_2([0, +\infty)) = [1, 2^{1-\alpha}).$$

The model represents a situation in which the medium 1 is the main advertising instrument, which is necessary to raise the goodwill level, whereas the medium 2 is a supporting medium, which enhances the main medium effect, but has no effect on the goodwill if used alone.

The function (26) is monotonically increasing and concave (see [1, p. 163, Corollary 5.19]). Again Φ has decreasing or constant returns to scale. Then Theorems 2-3 hold, in addition to Theorem 1, and we have that there exists an optimal solution u^* which determines the goodwill value $G^* = \prod_{i=1}^n \varphi_i(u_i^*) + G^0$, such that for all $i = 1, \dots, n$, either the condition (22) or (23) holds.

EXAMPLE 3 Assume quadratic costs, $c_i(u_i) = k_i u_i^2/2$, $i = 1, \dots, n$, and linear demand, $f(G) = \beta G$. If an optimal solution u^* has both coordinates positive, $u_1^* > 0$, $u_2^* > 0$, then it is characterized by the necessary and sufficient conditions (22), i.e.

$$\begin{aligned} \pi\beta\alpha\gamma_1 u_1^{\alpha-1} \left(\frac{1+2u_2}{1+u_2} \right)^{1-\alpha} &= k_1 u_1, \\ \pi\beta(1-\alpha)\gamma_1 u_1^\alpha \left(\frac{1+2u_2}{1+u_2} \right)^{-\alpha} \frac{1}{(1+u_2)^2} &= k_2 u_2, \end{aligned}$$

which has a unique solution.

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